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COMPUTATIONAL BUCKLING STRENGTH ANALYSIS OF STIFFENED PLATES WITH VARYING THICKNESS.

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Abstract. *Buckling of stiffened plates with variable plate thickness are studied. The main objective of the present paper is to present and verify an approximate, semi-analytical computational model applied to simply supported, stiffened plates with stepwise constant plate thickness and with in-plane loading. The method allows for a very efficient analysis of the structural response and the subsequent design of such plates. The deflections are represented by trigonometric functions. All combinations of biaxial in-plane compression or tension are included in the formulations. Estimation of the ultimate (buckling) strength is made in a simplified manner, using the von Mises yield criterion for the membrane stress as the strength criterion. The formulations derived are implemented in a Fortran computer code, and numerical results are obtained for a variety of plate and stiffener geometries. Relatively high numerical accuracy is achieved with low computational efforts. The results are, in most cases, found to be conservative compared to finite element analysis results. The model does not account for the reserve strength beyond the elastic buckling load.*

1 INTRODUCTION

Explicit design formulas [1,2] have traditionally been used to provide strength estimates of stiffened plates. These formulas are relatively simple to use, but their applicability is normally limited to plates with regular stiffener orientations and constant plate thickness etc. For other cases, for instance for plates with variable thickness and arbitrary stiffener orientations, other methods are required in order to obtain a structure that is able to sustain all loads and deformations with a suitable margin of safety. As an alternative, the finite element method could have been used, but this method is still impractical and too time consuming for most design purposes at present.

The main objective of the work is to develop a model for analysis of a stiffened plate with stepwise varying plate thickness using a semi-analytical method. The applicability of the model is verified by comparisons with fully nonlinear finite element analysis results. The work represents an extension of previous work [3,4] dealing with constant thickness plates with arbitrary stiffener orientations.

2 COMPUTATIONAL MODEL AND PLATE DEFINITION

The computations are carried out in two major steps. In the first step, the elastic buckling load (eigenvalue) and corresponding buckling mode of the stiffened plate are calculated. In the second step, a conservative buckling load (capacity) assessment is made of the plate, with a specified out-of-plane imperfection, using the von Mises yield criterion for the membrane stresses. In this computation, the imperfection shape is taken equal to the computed buckling mode, and its maximum value is specified, for instance in accordance with relevant codes or regulations. Displacements, for increasing loads, are computed using an approximate displacement magnifier involving the elastic buckling load.

The plate considered, Fig. 1, is simply supported with stepwise constant plate thickness, and it is subjected to in-plane biaxial compression or tension forces giving the mean stress values along the plate edges, as shown in the figure. The plate may have none, one or more symmetrical stiffeners. The stiffener orientations may be arbitrary, but in this paper, computations are restricted to plates with regular stiffener orientation. A stiffened plate is usually a part of a larger structure, and the edges are therefore forced to remain straight, which is the response assumed here. This will impose additional stress along the edges, but the mean values will not be affected. Further, the stiffeners are assumed to be sniped at their ends, and only their out-of plane bending (beam) stiffness is included in strain energy. Thus, their axial stiffness and its influence on the internal membrane stress distribution is neglected.

3 ELASTIC BUCKLING LIMIT STATE (ELS)

3.1 Material law and kinematic compatibility

For thin plates it is usual to assume plane stress conditions, in which case Hooke's law for an isotropic material reduces to

$$\sigma_x = \frac{E}{1-\nu^2}(\epsilon_x + \nu\epsilon_y) \tag{1}$$

$$\sigma_y = \frac{E}{1-\nu^2}(\epsilon_y + \nu\epsilon_x) \tag{2}$$

$$\tau_{xy} = \frac{E}{2(1+\nu)}\gamma_{xy} = G\gamma_{xy} \tag{3}$$

where σ_x, σ_y and τ_{xy} are the in-plane stresses, and ϵ_x, ϵ_y and γ_{xy} the in-plane strains. The material coefficients E and ν are the Young's modulus and the Poison's ratio, respectively. The total strain can be divided into a membrane strain and a bending strain:

$$\epsilon_x = \epsilon_x^m + \epsilon_x^b = \epsilon_x^m - zw_{,xx} \tag{4}$$

$$\epsilon_y = \epsilon_y^m + \epsilon_x^b = \epsilon_y^m - zw_{,yy} \tag{5}$$

$$\gamma_{xy} = \gamma_{xy}^m + \gamma_{xy}^b = \gamma_{xy}^m - 2zw_{,xy} \tag{6}$$

where w is the out-of-plane displacement in z-direction. Here, the bending strain contribution is accounted for by using Kirchoff's section assumption[5]. In large deflection theory, the membrane

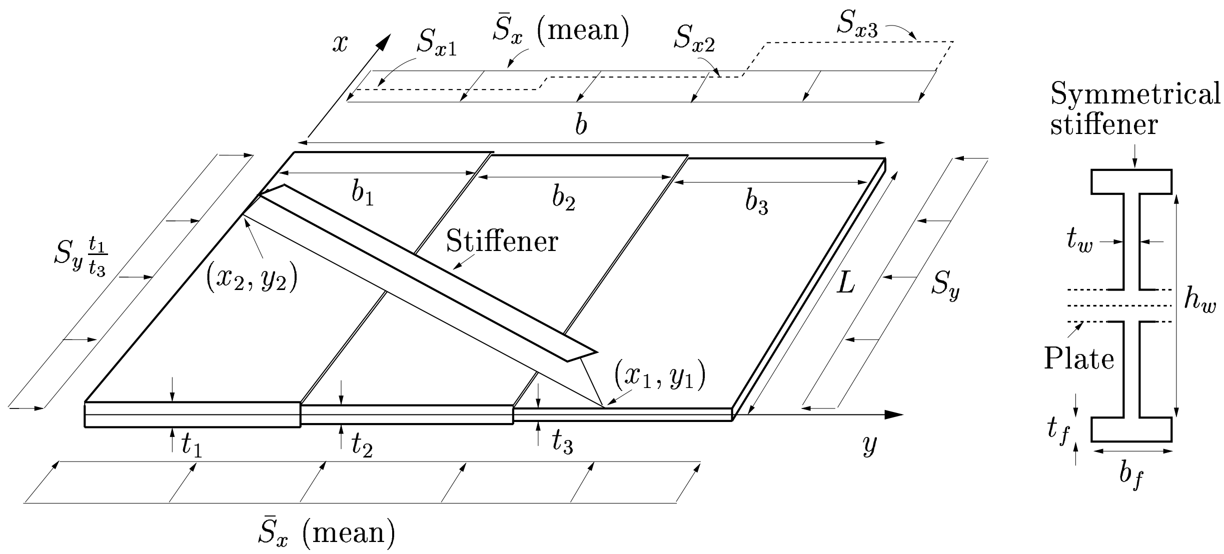


Figure 1: Simply supported plate with variable plate thickness and stiffener with end coordinates (x_1, y_1) and (x_2, y_2). $\bar{S}_x$ and S_y are the mean values of the applied, compressive stresses in x- and y-direction, respectively. The stiffener profile is symmetric about the middle plane.

strains in a plate with an initial imperfection w_0, in addition to w, can be written as:

$$\epsilon_x^m = u_{,x}^m + \frac{1}{2}w_{,x}^2 + w_{0,x}w_{,x} \tag{7}$$

$$\epsilon_y^m = v_{,y}^m + \frac{1}{2}w_{,y}^2 + w_{0,y}w_{,y} \tag{8}$$

$$\gamma_{xy}^m = u_{,y}^m + v_{,x}^m + w_{,x}w_{,y} + w_{0,x}w_{,y} + w_{0,y}w_{,x} \tag{9}$$

where u and v are the displacements in x- and y-direction, respectively. These, with imperfections included, were given by Marguerre [6], and represent an extension of von Karman's plate theory. In a linear elastic buckling analysis, the imperfections w_0 is set to zero.

3.2 Initial stresses

To determine the initial membrane stresses σ_x^{in} and σ_y^{in}, due to a reference loading ($\bar{S}_x^{in}$ and S_y^{in}), a linear static analysis of a perfect plate is performed. The edges are free to move in the in-plane directions, but forced to remain straight. If the plate consists of three plate strips with four degrees of freedom, Fig. 2, this assumption leads to the axial stiffness relationship

$$\frac{E}{1-\nu^2}\begin{bmatrix} (\frac{t_1 L}{b_1} + \frac{t_2 L}{b_2}) & -\frac{t_2 L}{b_2} & 0 & \nu(t_1 - t_2) \\ -\frac{t_2 L}{b_2} & (\frac{t_2 L}{b_2} + \frac{t_3 L}{b_3}) & -\frac{t_3 L}{b_3} & \nu(t_2 - t_3) \\ 0 & -\frac{t_3 L}{b_3} & \frac{t_3 L}{b_3} & \nu t_3 \\ \nu(t_1 - t_2) & \nu(t_2 - t_3) & \nu t_3 & (\frac{t_1 b_1}{L} + \frac{t_2 b_2}{L} + \frac{t_3 b_3}{L}) \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ P_y \\ P_x \end{bmatrix} \tag{10}$$

where $P_x = -\bar{S}_x^{in}(b_1 t_1 + b_2 t_2 + b_3 t_3)$ and $P_y = -S_y^{in} L t_3$ are the reference forces acting on the edges. Once Eq. 10 is solved, strains can be computed and inserted into Hooke's law, which gives the initial stresses in the plate stripes. These stresses are used in the geometrical stiffness matrix of the eigenvalue problem, and in the buckling strength (capacity) limit state (BLS) calculation presented below.

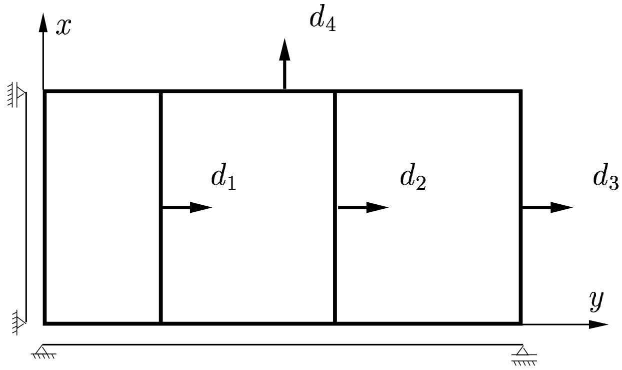


Figure 2: Definition of the in-plane degrees of freedom used in the linear static analysis, for a plate consisting of three plate strips with different thicknesses.

3.3 Eigenvalue problem

Ideal elastic buckling loads (eigenvalues) of a perfect, stiffened plate are computed using the Rayleigh-Ritz method. The assumed displacement field, which satisfies the boundary conditions of the plate, Fig. 1, is given by

$$w(x,y) = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}\sin(\frac{\pi i x}{L})\sin(\frac{\pi j y}{b}) \qquad \text{where} \begin{array}{ccc} 0 & \leq x \leq & L \\ 0 & \leq y \leq & b \end{array} \tag{11}$$

where a_{ij} are amplitudes, L is the plate length and b is the plate width. The total plate consists of several strips with different plate thickness. The first step is now to establish the potential energy of the plate, $\Pi = U + T$, where U is the strain energy and T is the potential energy of the external loads. Equilibrium requires that Π has a stationary value, $\delta\Pi = 0$. This requirement leads to displacements that must satisfy the matrix equation

$$(K^M_{ijkl} + \Lambda^e K^G_{ijkl})da^e_{kl} = 0 \quad \text{where} \quad K^M_{ijkl} = \frac{\partial^2 U}{\partial a_{ij}\partial a_{kl}} \text{ and } K^G_{ijkl} = \frac{\partial^2 T}{\partial a_{ij}\partial a_{kl}} \tag{12}$$

Here, K^M_{ijkl} is the material stiffness matrix, K^G_{ijkl} the geometrical stiffness matrix, Λ^e the eigenvalues and da^e_{kl} the eigenvectors.

The elastic strain energy contribution from bending of the plate can now be given by

$$U^b_{\text{plate}} = \int_0^b\int_0^L \frac{Et^3(y)}{24(1-\nu^2)}[(w_{,xx}+w_{,yy})^2 - 2(1-\nu)(w_{,xx}w_{,yy} - w^2_{,xy})]\,dx\,dy \tag{13}$$

The membrane strain energy is not included as it does not affect computed eigenvalues.

The displacements of the stiffeners are equal to the displacements in the plate along the stiffener. In the present model, the stiffeners are accounted for by using ordinary beam theory. For each stiffener, this gives a bending strain energy defined by

$$U^b_{\text{stiffener}} = \frac{1}{2}\int_S EI_e\left(\frac{(x_2-x_1)^2 w_{,xx} + 2(x_2-x_1)(y_2-y_1)w_{,xy} + (y_2-y_1)^2 w_{,yy}}{(x_2-x_1)^2+(y_2-y_1)^2}\right)^2 dS \tag{14}$$

where S is the length and I_e the effective moment of inertia of the stiffener with end coordinates (x_1, y_1) and (x_2, y_2).

Finally, the potential energy of external loads due to plate bending is given by

$$T = \int_0^L \int_0^b \frac{t(y)}{2}(\sigma_x^{in} w_{,x}^2 + \sigma_y^{in} w_{,y}^2)\, dy\, dx \tag{15}$$

where σ_x^{in} and σ_y^{in} are the initial membrane stresses, found in the linear static analysis.

4 BUCKLING STRENGTH LIMIT STATE (BLS)

4.1 Strain compatibility

For internal stress computations, a plate with an imperfection w_0 is considered. In order to account for the stress redistribution due to an additional out-of-plane displacement w, it is necessary to obtain an equation which represents in-plane strain compatibility. This can be done by differentiation and combination of Eqs. 7-9, which gives the compatibility equation

$$\epsilon_{x,yy}^m + \epsilon_{y,xx}^m - \gamma_{xy,xy}^m = w_{,xw}^2 - w_{,xx}w_{,yy} + 2w_{0,xy}w_{,xy} - w_{0,xx}w_{,yy} - w_{0,yy}w_{,xx} \tag{16}$$

By substituting strains from Hooke's law and Airy's stress function $F(x, y)$, defined by

$$\sigma_x^m = F_{,yy} \qquad \sigma_y^m = F_{,xx} \qquad \tau_{xy}^m = -F_{,xy} \tag{17}$$

into this equation, the following nonlinear plate compatibility equation (Marguerre[6]) is obtained

$$\nabla^4 F = E(w_{,xw}^2 - w_{,xx}w_{,yy} + 2w_{0,xy}w_{,xy} - w_{0,xx}w_{,yy} - w_{0,yy}w_{,xx}) \tag{18}$$

With the assumed displacement field w and initial imperfection w_0, the solution of Eq. 18 can be written as

$$F(x, y) = \Lambda(\frac{1}{2}\sigma_x^{in} y^2 + \frac{1}{2}\sigma_y^{in} x^2) + \sum_{i=0}^{2m}\sum_{j=0}^{2n} f_{ij} \cos(\frac{i\pi}{L}x)\cos(\frac{j\pi}{b}y) \tag{19}$$

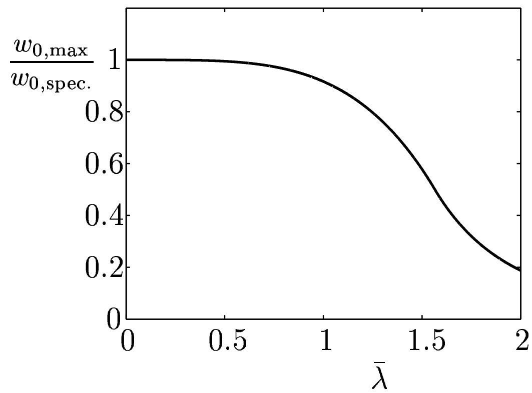
where Λ is the load factor at the current load step and the coefficients f_{ij} are functions of w and w_0. Details of these coefficients can be found in the literature[7]. The membrane stresses in the plate at each load step, can now be calculated from Eq. 17. In the present work, the imperfection displacement field w_0 is the first eigenmode of the eigenvalue problem defined by Eq. 12.

4.2 Load incrementation

In linearized elastic second order theory, the displacement w beyond the initial displacement imperfection (w_0) can be estimated, using an approximate displacement magnifier[8], from

$$w = \frac{\Lambda}{\Lambda_{cr}^e - \Lambda} w_0 \tag{20}$$

where Λ_{cr}^e is the load factor of the first eigenvalue. This expression is used in this paper, for an increasing load factor Λ, to estimate a buckling strength (capacity) using first yield of the von Mises' membrane stress as the strength criterion. Use of Eq. 20 implies that this strength estimate will never exceed the elastic buckling load. The model is consequently conservative as it is not able to capture the reserve strength beyond this load.



$$\frac{w_{0,\max}}{w_{0,\text{spec.}}} = \begin{cases} (1 - \frac{1}{12}\bar{\lambda}^4) & , \text{if } \bar{\lambda} \leq \sqrt{1.56} \\ 3/\bar{\lambda}^4 & , \text{if } \bar{\lambda} \geq \sqrt{1.56} \end{cases}$$

Figure 3: The calibrated imperfection amplitude $w_{0,\max}$ relative to the amplitude $w_{0,\text{spec.}}$ specified by relevant rules.

4.3 Imperfection amplitude

The conservativeness mentioned above, that generally increases with increasing slenderness, may partly be reduced by adopting a slenderness dependent maximum imperfection amplitude $w_{0,\max}$. A proposed variation, in terms of the reduced slenderness

$$\bar{\lambda} = \sqrt{\frac{\Lambda_Y}{\Lambda_{cr}^e}} \tag{21}$$

is given in Fig. 3. There $w_{0,\text{spec.}}$ is a specified imperfection amplitude that in a design situation is to be taken equal to a relevant design code[1, 2]. The slenderness dependency of the proposal in Fig. 3 has been chosen following comparisons with fully nonlinear element analysis results and with relevant codes when applicable.

5 RESULTS

5.1 Representation of the results

Based on the theory in the present paper, a Fortran computer code was developed. Results computed by the present model have been compared to finite element analyses using Ansys for a variety of plate and stiffener dimensions. The element mesh used in Ansys is very fine, as shown in Fig 6 (right), where 1860 eight-node elements elements are used. In comparisons, analyses calculated by the present model are performed with 225 degrees of freedom (15x15).

First, calculations of the elastic buckling limit state (ELS) are examined. Then, calculated buckling strengths (BLS) are examined for plates with regular stiffeners and stepwise constant plate thickness. Results are presented for various combinations of edge loadings $\bar{S}_x$ and S_y, which are given by the initial reference values ($\bar{S}_x^{in}$ and S_y^{in}) times the relevant load factor.

5.2 Verification of the elastic buckling limit state (ELS)

Fig. 4 shows interaction curves for the elastic buckling limit state of two unstiffened plates with plate dimensions defined in Table 1. The material properties are $E = 208000MPa$ and $\nu = 0.3$. The agreement with finite element analysis is generally seen to be very good.

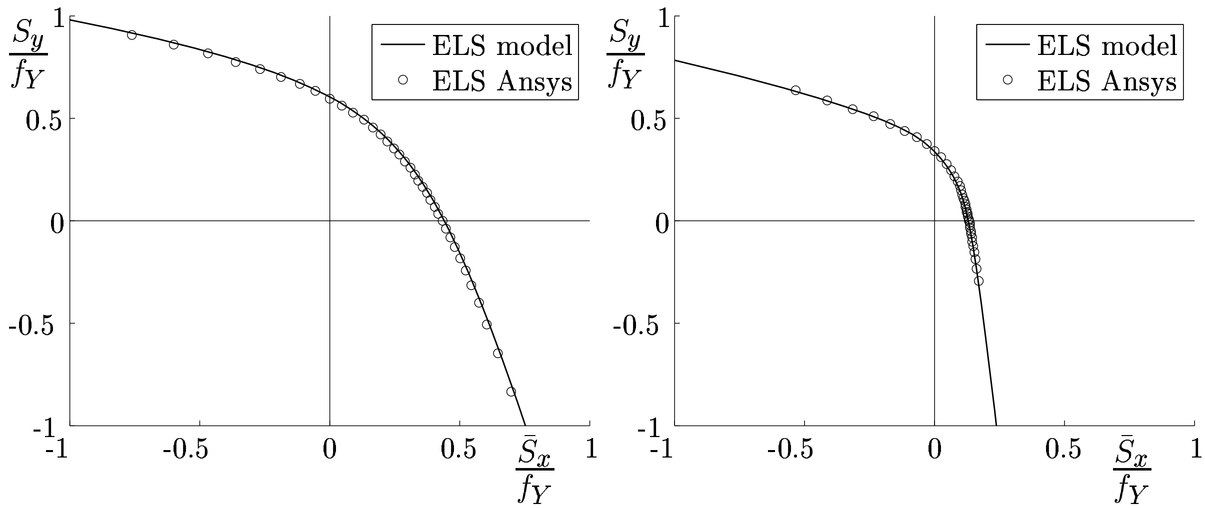


Figure 4: Comparisons of present model with Ansys of plate 1 (left) and 2 (right), as defined in Table 1.

Table 1: Dimension $[mm]$ of the unstiffened plates in the ELS comparisons.

	L	b	b_1	b_2	b_3	t_1	t_2	t_3
Plate 1	1000	1500	500	500	500	20	20	10
Plate 2	1000	3000	1000	1000	1000	14	12	10

Additional comparisons of ELS predictions of the present model and finite element analysis results are shown in Fig. 7 and 8. The discrepancy between the two relevant curves is caused by the assumptions made for the stiffeners in the present model, where torsional deflection and axial membrane stiffness of the stiffeners are neglected.

5.3 Verification of the buckling strength (BLS) predictions

In this section, the present buckling strength limit state results (BLS) are compared with ultimate limit state results (ULS) obtained from fully nonlinear finite element analyses using Ansys. The ULS is reached when the external loads reach a maximum value, and the structure becomes unstable.

Results are presented for four plates with stepwise variable plate thickness and two stiffeners with regular orientation. The dimensions of the plates are given in Fig. 5. As shown in Fig. 1 (right), the stiffeners are symmetric about the middle plane. Conforming to DNV rules [2], the specified imperfection is taken as $w_{0,\text{spec.}} = L/(3\text{x}200) = 5mm$. The material properties are Young's modulus $E = 208000MPa$, yield strength $f_Y = 235MPa$ and Poison's ratio $\nu = 0.3$. The Ansys analyses are performed with a hardening modulus $E_T = 1000MPa$ and $w_{0,\max} = w_{0,\text{spec.}}$. Fig. 6 shows the buckling mode of plate 3 subjected by uniaxial mean stress $\bar{S}_x$, calculated by the present model (left) and by Ansys (right). The agreement is seen to be good. This buckling mode can be considered as a local buckling mode, as the lateral displacement of the plate along the stiffener is small.

In Fig. 7 and 8, presenting interaction curves of the plates, the BLS-curves have to be compared with the ULS-curves. Depending on the load combination, the reduced slenderness

	L	b	t_1	t_2	t_3	h_f	t_f	b_f	t_f
Plate 3	3000	2100	30	28	26	300	10	150	20
Plate 4	3000	3300	30	28	26	300	10	150	20
Plate 5	3000	2100	24	22	20	300	10	150	20
Plate 6	3000	2100	16	14	12	300	8	100	10

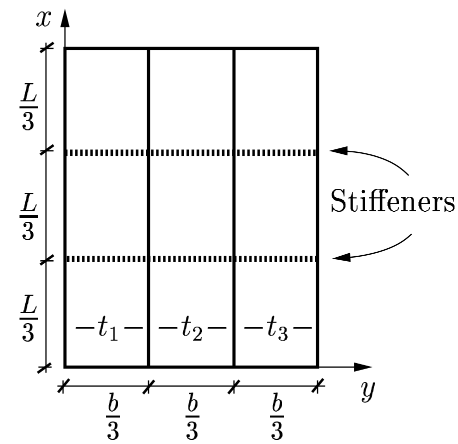


Figure 5: Overview and dimensions [mm] of the plates with stepwise constant plate thickness and two stiffeners with regular orientation.

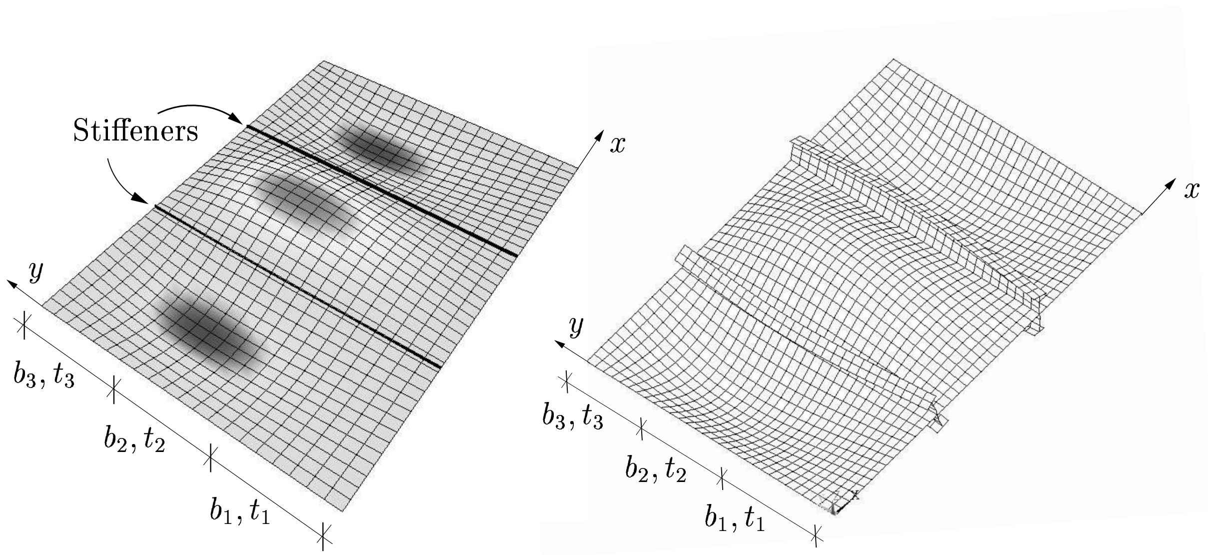


Figure 6: The buckling modes of plate 3 subjected to an uniaxial external stress $\bar{S}_x$, calculated by the present model (left) and Ansys (right)

varies along the interaction curves. As shown in Fig. 3, the imperfection amplitude used in the present BLS model is reduced for slender plates, because these plates possess a reserve strength beyond the linear elastic buckling load. For instance for plate 3, the maximum value of the reduced slenderness is approximately 1.17 ($\bar{S}_x = 1.67 S_y$). This gives an imperfection amplitude $w_{0,\max} = 0.84 w_{0,\text{spec.}}$. The agreement between the present model and Ansys is good for such a small reduced slenderness. For plate 6, the maximum value of the reduced slenderness is approximately 2.36 ($\bar{S}_x = 1.43 S_y$), which gives an imperfection amplitude $w_{0,\max} = 0.10 w_{0,\text{spec.}}$. This imperfection amplitude is physically speaking, unreasonable small, but the calculated BLS strength results are still conservative compared to the full nonlinear finite element analysis results. The ULS capacity predicted by Ansys is in this case, twice as large as the BLS strength calculated by present model. The “squash yield state” according to the von Mises yield criterion are also included in the figures, for the sake of illustration.

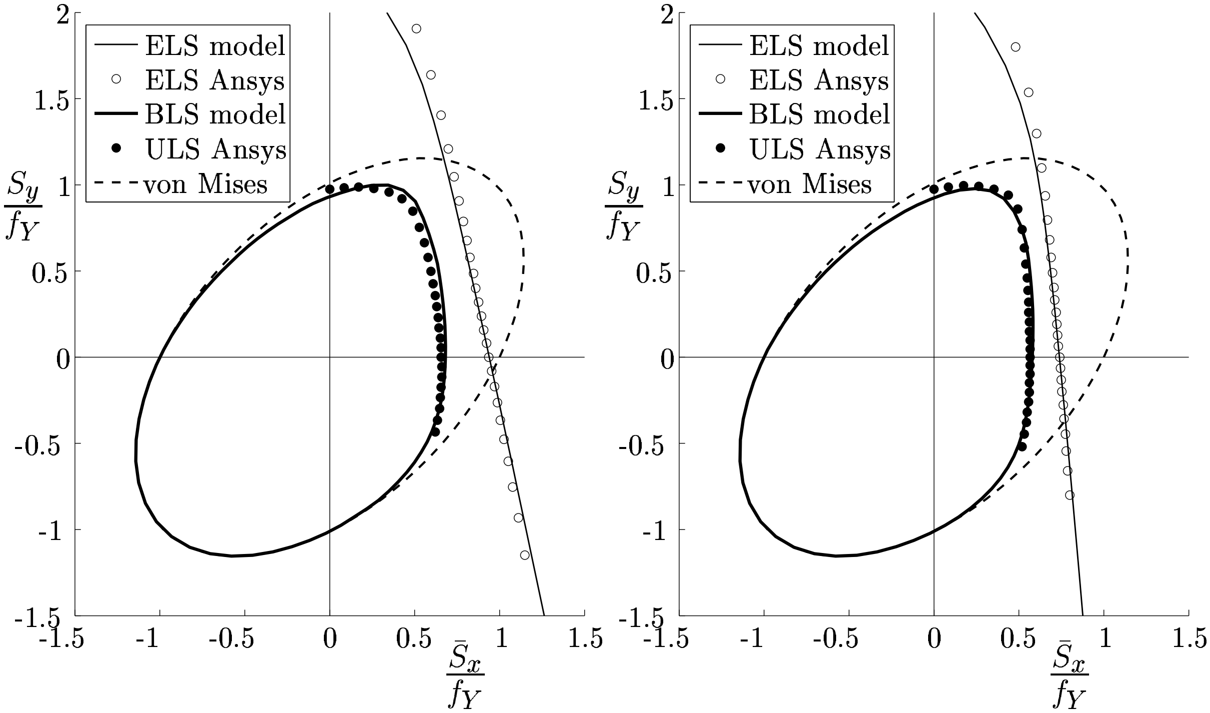


Figure 7: Interaction curves for plate 3 (left) and 4 (right), defined in Fig. 5

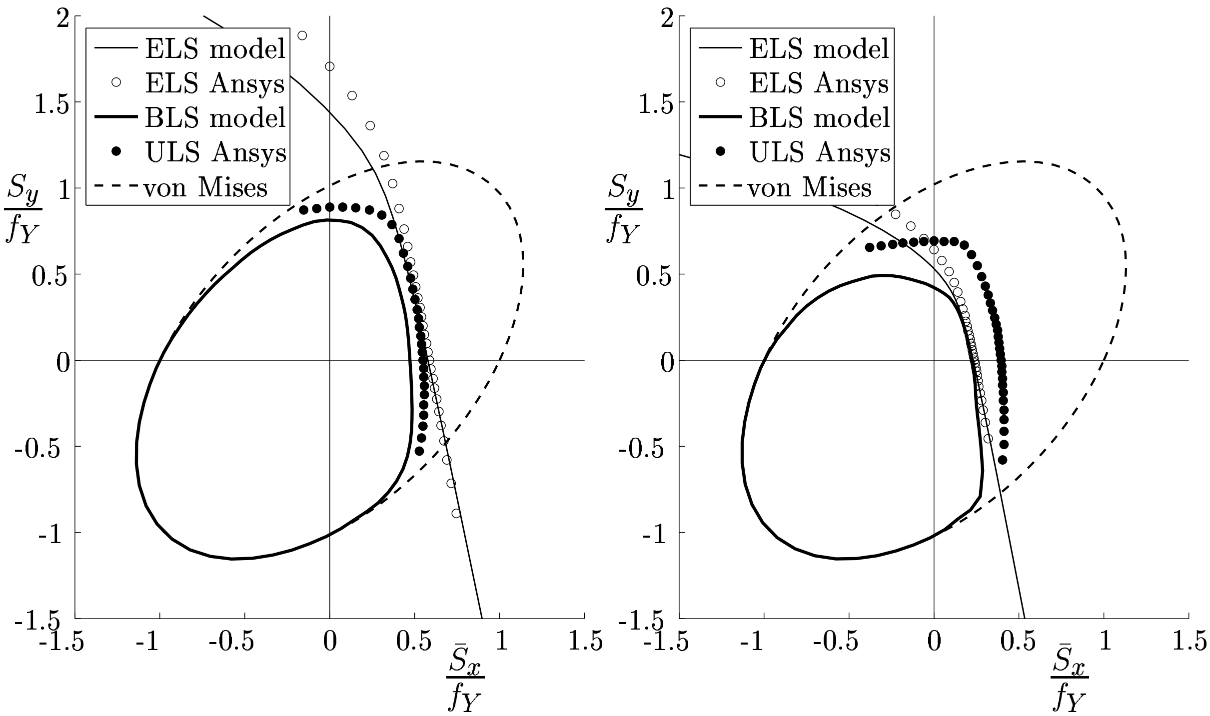


Figure 8: Interaction curves for plate 5 (left) and 6 (right), defined in Fig. 5

6 CONCLUDING REMARKS

An efficient computational model for elastic buckling analysis of a plate with stepwise constant plate thickness and regular stiffener orientations is presented. Good agreement with finite element analysis of the elastic buckling stress is achieved, and the results are in most cases conservative compared to finite element analysis results. The buckling strength (capacity estimates) of a plate with imperfections is estimated using first yield as strength criterion. The model does not account for the reserve strength after the elastic buckling stress is reached. To extend the present method to account for this strength, displacements must be computed using large displacement plate theory [7]. This requires further work. Det Norske Veritas (DNV) has implemented the present computational model, without option for thickness variation, into their computerized buckling code PULS (Panel Ultimate Limit State) [9].

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